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New Reductions and Nonlinear Systems for the 2D Schrodinger Operators

Abstract. *New Completely Integrable $(2 + 1)$ -System is studied. It is based on the so-called $L - A - B$ -triples $L_t = [H, L] - fL$ where L is a 2D Schrodinger Operator. This approach was invented by S.Manakov and B.Dubrovin, I.Krichever, S.Novikov(DKN) in the works published in 1976¹. A nonstandard reduction for the 2D Schrodinger Operator (completely different from the one found by S.Novikov and A. Veselov in 1984) compatible with time dynamics of the new Nonlinear System, is studied here. It can be naturally treated as a 2D extension of the famous Burgers System. The Algebro-Geometric (AG) Periodic Solutions here are very specific and unusual (for general and reduced cases). The reduced system is linearizable like Burgers. However, the general one (and probably the reduced one also) certainly lead in the stationary AG case to the nonstandard examples of algebraic curves $\Gamma \subset W$ in the full complex 2D manifold of Bloch-Floquet functions W for the periodic elliptic 2D operator H where $H\psi(x, y, P) = \lambda(P)\psi(x, y, P)$, $P \in \Gamma$. However, in the nontrivial cases the operators are nonselfadjoint. A Conjecture is formulated that for the nontrivial selfadjoint elliptic 2D Schrodinger operators H with periodic coefficients **The Whole 2D Complex Manifolds W cannot not contain any Zariski open part of algebraic curve Γ except maybe one selected level $H\psi = \text{const}$ found in 1976 by DKN.** This version contains new results. It also corrects some non-accurate claims; in particular, the non-reduced system is non-linearizable in any trivial sense.*

¹It was developed later by many authors. Specific case with 2nd order Schrodinger Operator H gauge equivalent in fact to our new system over complex field, appeared first time as the initial example of S.Manakov (who considered this system uninteresting, publishing it under the pressure of S.Novikov for justifying the idea: What is a right 2D analog of Lax pairs?) No investigation of it was performed later, it never was mentioned in the later literature. No reductions, no theory of special solutions. No applications of this specific system to the inverse spectral theory for 2D elliptic self-adjoint operators with data borrowed from one energy level, has been made (which was a main program of DKN started in 1976)– probably because no nontrivial self-adjoint reductions are known for this system even now. By this reason we call this system and its reductions GMMN-Grinevich, Mironov, Manakov, Novikov, reserving for the reduction below–see the case III– notation B_2 -system as a ”2D Burgers”.

1. 2D Schrodinger Operators and Nonlinear Systems. The new system.

As it was pointed out in the work [1] and developed in [2], the right $2+1$ analog of $1+1$ Lax Pairs $L_t = [H, L]$ for the nontrivial second order Schrodinger Operators $L = \Delta + G'\partial_x + G\partial_y + S$ is

$$L_t = [H, L] - fL$$

We always will use gauge condition $G' = 0$. We call them $L - A - B$ -triples where $A = H$ and $B = f$ in this notations. Such equation is equivalent to the equations

$$(L_t - [H, L])\psi = 0$$

for all ψ such that $L\psi = 0$. A lot of works was written by the Moscow Soliton group about these systems and **Inverse Spectral (Scattering) Problems for the elliptic 2D Schrodinger operators L with periodic and rapidly decreasing coefficients based on the data collected from One Energy Level only.** (see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]). Special attention was paid to the **Purely Potential Reduction $G = 0$** . It was found effectively in [5, 6] and leads to the Prym's Θ -functions in the periodic case. Corresponding Nonlinear Systems (the NV-Hierarchy) always have operators H of odd order in each variable. In particular, the first nontrivial operators H have order 3 in both variables x, y and present a 2D extension of the famous KdV system different from KP.

We consider in this work the system $L_t = [H, L] - fL$ where

$$L = \partial_x\partial_y + G\partial_y + S, H = \partial_x^2 + \partial_y^2 + F\partial_y + A$$

Proposition 1. *Following GMMN system of evolutional equations follows from the $(L - H - f)$ -triple:*

$$G_t = G_{xx} - G_{yy} + (F^2/4)_x - (G^2)_x - A_x + 2S_y$$

$$S_t = S_{yy} - S_{xx} - 2(GS)_x + (FS)_y$$

with differential constraints

$$F_x = 2G_y, A_y = 2S_x, f = 2G_x - F_y$$

The reduction $S = 0$ is well-defined. It is a special 2D extension of the famous Burgers system. So we call it B_2 .

Proposition 2. *The elementary substitution:*

$$G = -(\log c)_x, F = -2(\log c)_y$$

$$A = -2u_x, S = -u_y$$

leads to the form

$$[c^{-1}(c_t - c_{xx} + c_{yy})]_x = -2(u_{xx} - u_{yy})$$

$$(u_t + u_{xx} - u_{yy} + 2u_y c_y/c)_y = 2(u_y c_x/c)_x$$

The condition $S = u_y = 0$ is time invariant, and the B_2 system reduces to the linearized form:

$$c_t - c_{xx} + c_{yy} = (\Phi(y, t) + A(x, t))c$$

So the reduced system B_2 is linearizable in the variable c as the ordinary Burgers corresponding to the y independent solutions of the B_2 system.

Both statements can be easily checked by direct calculation.

Corollary 1. Every product function $c_i = c'_i(x, t)c''_i(y, t)$ satisfying to the pair of separate equations in the variables x and y

$$c'_{it} = c'_{ixx} + A(x, t)c'_i, c''_{it} = -c''_{iyy} + \Phi(y, t)c''_{iyy}$$

satisfies to the full equation. Every linear combination of such solutions $c = \sum_i c'_i(x, t)c''_i(y, t)$ satisfies to the full equation for c leading to the family of coefficients $F = -2c_x/c$ and $G = -c_y/c$ nontrivially depending of all variables.

How to describe effectively the reduction $S = 0$ for the Bloch functions of the AG 2D Schrodinger Operator $L = \partial_x \partial_y + G \partial_y + S$? How to find Algebro-Geometric Solutions to the linear equation $L\psi = 0$ and nonlinear equation $L_t = [H, L] - fL$?

2. The Algebro-Geometric Solutions.

Let us start with nonsingular Algebraic Curve Γ accompanied by the standard Inverse Spectral Data consisting of 2 infinite points P_1, P_2 with local parameters $z_1 = 1/k_1, z_2 = 1/k_2$ correspondingly, and with selected generic set of g points $D = Q_1 + \dots + Q_g$. We construct a function $\psi(x, y, t, P)$ on Γ depending on parameters x, y, t such that:

1. ψ is meromorphic on Γ except the points P_1, P_2 where it has asymptotic

$$P_1 : \psi = c_1(x, y, t) \exp\{k_1 x + k_1^2 t\} (1 + u/k_1 + \dots)$$

$$P_2 : \psi = c_2(x, y, t) \exp\{k_2 y + k_2^2 t\} (1 + a/k_2 + \dots)$$

2. ψ has exactly g poles of the first order whose position is independent of parameters x, y, t .

3. ψ should be normalized. We use the standard normalization condition $\psi(0, 0, 0, P) = 1$ and $c_1 = 1$ for the general operator L with gauge condition $G' = 0$ (here may be $S \neq 0$). It is well-known that in order to choose another gauge condition $S = 0$ for the general case, i.e. $L = \partial_x \partial_y + G' \partial_x + G \partial_y$, we need to use the "condition K_0 ": $\psi(x, y, t, P_0) = 1$ fixing any arbitrary selected point $P_0 \in \Gamma$ distinct from D, P_1, P_2 .

An extension of this approach will be used for the most important case III containing the main result of the present work. We are going to use in the case III below such normalization of ψ that additional k points and value of ψ in all of them are fixed (we call it "the condition K_k "). In this case we need to take divisor D of the degree $g + k$. Such trick was used by Krichever few years ago in the works dedicated to orthogonal coordinates. We will modify it below and use for our goals. It has some weak points (f.i. it is noneffective in terms of formulas) but its version is absolutely necessary for us.

As usually, we obtain 2 following statements:

Statement 1. *The functions ψ satisfy to the equations*

$$(\partial_t - H)\psi = 0, L\psi = 0$$

with P -independent coefficients

$$c = c_2, c_1 = 1, G = -c_x/c, S = -u_y = -a_x + G_y, F = -2c_y/c$$

$$A = -2u_x = -2a_y + 2c_y^2/c^2 + c_t/c - \Delta c/c$$

Statement 2. *Let an algebraic function $\lambda(P)$ exists on Γ such that it has only 2 poles of second order in the points P_1, P_2 with principal parts equal exactly to k_1^2, k_2^2 . Then $H\psi = \lambda\psi$ for all $P \in \Gamma$.*

Anyway, the compatibility requirement leads to the nonlinear equation above $L_t = [H, L] - fH$. In the second case we have $L_t = 0$.

There are following cases:

Case I: No restriction, Standard Normalization, Nonsingular Riemann Surface. We obtain the algebro-geometric solutions to the generic system above. Their reality condition (all coefficients F, G, S, A are real) can be easily found:

Lemma 1. 1. Let nonsingular Riemann Surfaces of finite genus with antiholomorphic involution τ be given such that

$$\tau(P_j) = P_j, \tau^*(k_j) = -\bar{k}_j, \tau(D) = D$$

Then all coefficients F, G, S, A are real. 2. Let also an algebraic function λ be given such that it has only second order poles in the points P_1, P_2 with principal parts k_1^2, k_2^2 . Then $H\psi = \lambda\psi$ for the real elliptic second order operator $H = \Delta + F\partial_y + A$

Another interesting class is such that the operator H is self-adjointed. Is it possible? Formally, such class can be obtained from the same class of "real" Riemann surfaces and antiholomorphic involutions but we should take divisors D satisfying to the equation

$$D + \tau D \sim K + P_1 + P_2$$

(we call it "The Cherednik Type Equation"). However, this equation is not solvable for the nontrivial Riemann Surfaces. It is solvable for such cases that operator H has odd orders in both variables x, y . For the second order H it leads to the trivial class only.

Easy to see directly from the nonlinear equations that "physical" initial values are not invariant under the time dynamics. Here $iF_x/2 \in R$ and $eU = A - F_y/2 - F^2/4 \in R$ (they are called Magnetic Field eB and Electric Potential eU by physicists). Their reality is necessary for the operator H to be self-adjoint.

Conjecture. Consider any nontrivial self-adjoint smooth periodic (topologically trivial) second order operator

$$H = \partial_x^2 + (\partial_y + F/2)^2 + U = \Delta + F\partial_y + A$$

$$A = U - F^2/4 - F_y/2$$

(i.e. F and A both are periodic in x and in y with periods (T_1, T_2) . Construct its full 2D complex manifold W of all Bloch-Floquet Formal Eigenfunctions $\psi(x, y, P)$:

$$\psi(x + T_1, y, P) = \mu_1 \psi(x, y, P), \psi(x, y + T_2, P) = \mu_2 \psi(x, y, P)$$

$$H\psi = \lambda(P)\psi, P \in W$$

Let an algebraic nonsingular curve Γ be given such that its Zariski open part Γ^* is a complex submanifold in W ,

$$\Gamma^* \subset W, \Gamma = \Gamma^* \cup \infty$$

Here ∞ is a finite set of points. Then either the surface Γ^* coincides with some "energy" level $\lambda = \text{const}$ or coefficients of operator H can be reduced to the sums of functions of one variable.

Let us remind here that Feldman, Trubowitz and Knorrer proved in 1990s (see [15]) a "Novikov Conjecture" (formulated in early 1980s): For any real smooth periodic purely potential 2D Schrödinger Operator 2D its Complex Bloch-Floquet manifold W of eigenfunctions can be a Zariski open part of algebraic variety only if potential is equal to the sum of functions of one variable,

$$A = U = f(x') + g(y')$$

Here x', y' are orthonormal coordinates. Probably, even stronger statement is true formulated above and confirmed by the algebro-geometric examples.

Case II: Take purely potential NV-reduction on L where $c_1 = c_2 = c = 1$. Easy to prove that in this case we have $F = G = 0, S = g(x+y) + h(x-y)$, so this case leads to the trivial result: our new system is compatible with this reduction only for trivial operators. From the nonlinear system we deduce immediately that

$$A_x = 2S_y, A_y = 2S_x, F_x = 2G_y, F_y = 2G_x, f = 0$$

In the stationary case we need to have algebraic function λ with 2 poles only in the points P_1, P_2 with principal parts k_1^2, k_2^2 and holomorphic involution

$$\sigma : \Gamma \rightarrow \Gamma, \sigma^2 = 0$$

But we need also to have divisors D such that $D + \sigma(D) \sim K + P_1 + P_2$. Unfortunately, such divisors do not exist in the nontrivial cases where potential $A \neq f(x-y) + g(x+y)$ for the second order purely potential operators $H = \Delta + A$.

For that we need operators H of the odd orders in both variables x, y . The so-called NV-systems are always such that both these orders are equal to each other. Let us mention that Taimanov in [16] and recently Krichever

in [17] considered the whole hierarchy where orders can be different (for the investigations of "Novikov Conjecture" for the analog of Riemann-Shottki Problem for the Prym's Θ -functions).

Case III: Consider now our new reduction $S = 0$ (see above: we have $A(x, t)$ non necessary equal to zero here). How to find algebro-geometric solutions? Let us describe here spectral data corresponding to this case. Our picture can be viewed as a result of degeneration in the family of algebraic curves Γ_ϵ for $\epsilon \rightarrow 0$. In the final moment we obtain a group of transversal crossings to the points where small cycles degenerated. In our specific case the collection of vanishing cycles divides Riemann Surface Γ into 2 pieces $\Gamma = \Gamma' \cup \Gamma''$.

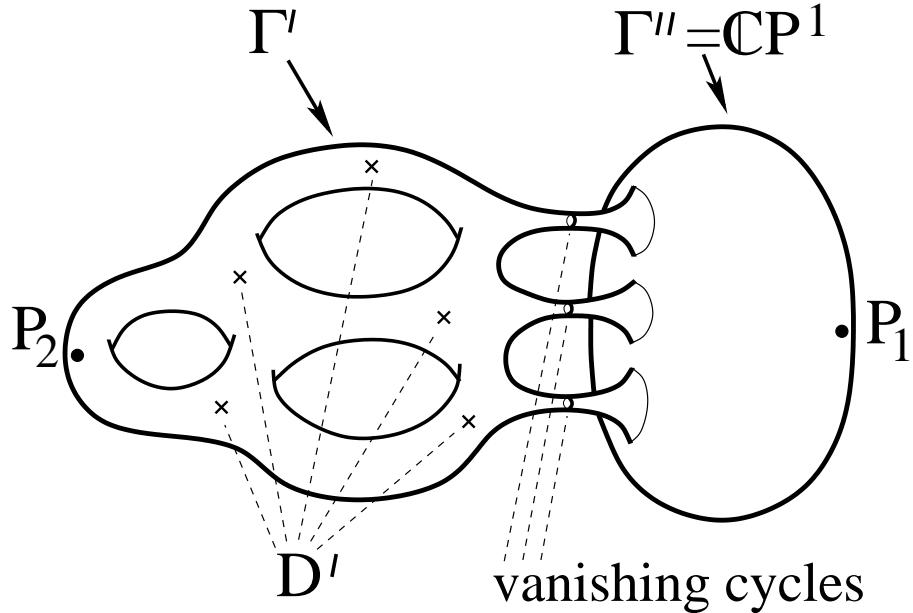


Fig 1. The case $A = 0$.

One of them Γ'' should have genus equal to zero in the case $A = \text{const}$. If $A(x)$ does not depend on t , we require it to be some finite-gap potential from KdV theory $w = A'(x)$, and Γ'' should be corresponding hyperelliptic Riemann surface with selected point $P_1 \in \Gamma''$.

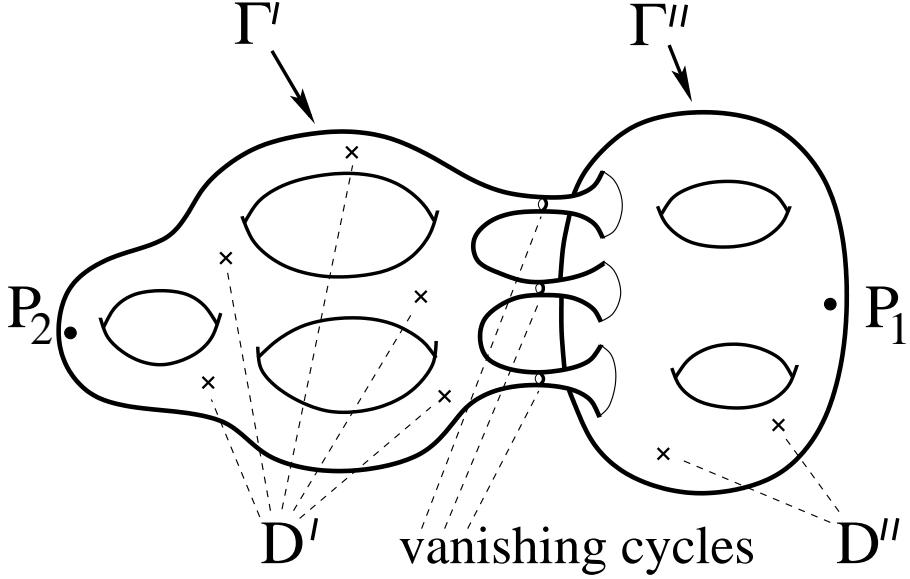


Fig 2. The case $A \neq 0$.

It is equal to the Weierstrass point with corresponding canonical local parameter. For the general finite-gap (AG) time-dependent (KP) case we take any nonsingular Riemann surface Γ'' with selected point P_1 and local parameter k_1 . We construct as usual a Baker-Akhiezer Krichever type function $\psi''(x, t, P)$ on Γ'' with divisor D'' and asymptotic

$$\exp\{k_1 x + k_1^2 t\}(1 + \dots)$$

near P_1 .

So we finally need to take degenerate Riemann surface Γ such that

$$\Gamma = \Gamma'' \bigcup \Gamma', P_1 = \infty \in \Gamma'', P_2 \in \Gamma'$$

Let genuses of these surfaces be g', g'' correspondingly. The intersection of them is a discrete set outside of infinities P_1, P_2 . We assume that it is finite. The Riemann Surface Γ' should be also algebraic. The union of this assumptions is called a "Finite-Gap Property" in our case. So we have:

$$\Gamma'' \cap \Gamma' = R = (R_0 \bigcup R_1 \bigcup \dots \bigcup R_k)$$

Here R is a union of $k+1$ distinct points R_q imbedded into the both Riemann surfaces $R \rightarrow R' \subset \Gamma' = S^2$ and $R \rightarrow R'' \subset \Gamma''$. We assume that both R', R''

do not meet infinite points P_2, P_1 and divisors D', D'' . The divisor D is completely concentrated in the part Γ' for the case $A = 0, \Gamma'' = CP^1$ where D'' is empty: there are no divisor points in $CP^1 = \Gamma''$. For the general case we have $D = D' \cup D''$ correspondingly in Γ', Γ'' of the degrees $g' + k$ for $D' \subset \Gamma'$ and g'' for $D'' \subset \Gamma''$.

Let us point out that the points $R'_s \in CP^1$ can be identified with the wave numbers p_s such that $p_0 = 0$. The simplest natural functions in this case are:

$$\phi_s(x, t) = \exp\{p_s x + p_s^2 t\}, \phi_0 = 1$$

In the case $A \neq 0$ these points in Γ'' define wave functions $\psi''(x, t, R''_s) = \phi_s(x, t)$

Remark. *We can take any solutions $\phi_s(x, t)$ to the equation*

$$(\partial_t - \partial_x^2) \phi_s(x, t) = 0, s = 1, \dots, k$$

instead of simple exponents for the case $\Gamma'' = CP^1$. We always require $\phi_0 = 1$. For the case $A \neq 0$ all our functions $\phi_s(x, t)$ satisfy to the equation

$$(\partial_t - \partial_x^2 - A(x, t)) \phi_s = 0$$

Theorem 1. *Let Γ' be a nonsingular algebraic curve, the point P_2, P_1 and divisor D' of degree $g' + k$ is generic. There exists an unique function $\psi'(x, y, t, P \in \Gamma')$ with following properties: It has exponential asymptotic near the point $P = P_2 \in \Gamma'$ as above*

$$k = k_2, \psi = c(x, y, t) \exp\{ky + k^2 t\} (1 + v(x, y, t)/k + \dots)$$

It has first order poles in the fixed points $D'_s, s = 1, 2, \dots, d + k$, and

$$\psi'(x, y, t, R''_s) = \phi_s(x, t), s = 0, 1, \dots, k$$

This functions satisfy to the equations (below) for all $P \in \Gamma'$

$$(\partial_t - H)\psi = 0, L\psi = 0$$

where

$$G_y - v_x = S = u_y = 0, A = -u_x = A(x, t), G = -c_x/c, F_x = 2G_y, F = -2c_y/c$$

$$G_t = (\partial_x^2 - \partial_y^2)G + (F^2/4 - G^2)_x$$

The proof is more or less standard. Except cancelation at infinities, we get also cancelation of values in all selected points which is possible only if $S = 0, A = A(x, t)$. It essentially implies our result unifying this property with the traditional arguments for this area. The important breakthrough for us was to find these data.

Corollary 2. Let a nonsingular Riemann surface Γ' be such that there exists a meromorphic function $\lambda(P)$ with only pole of second order in $P_2 \in \Gamma'$. It means that this surface is hyperelliptic $\mu^2 = P_{2g+1}(\lambda), P_2 = \infty$. Then coefficients of both operators do not depend on t and

$$H\psi = \lambda\psi(x, y, P), H = \Delta + F(x, y)\partial_y$$

Can these operators be trivial if we take arbitrary data?

In the next work we will clarify all this details and present more precise formulas. There is something strange in this linearizable "Burgers Style" Reduction $S = 0$: *Does it generate nontrivial examples for the Spectral Theory according to the present results or in fact these examples are in some sense trivial? We constructed them above but did not investigated them enough to prove this statement completely. We still have some doubts.*

The reduced equation can be viewed as a special Integrable 2D Extension of the famous Burgers Equation. The reduced system is linearizable, the full system is non-linearizable. So many solutions of it can be found completely elementary, in style of the ordinary Burgers equation. However, the stationary solutions for the AG cases possibly leads to the nontrivial examples of the Algebraic Curves $\Gamma' \in W$ such that $H\psi = \lambda\psi$ and $\lambda \neq \text{const}$. We know that for the full system $S \neq 0$ these type of examples certainly are nontrivial. Some of them for $g = 2$ were extracted by the authors from the analysis (made by Mironov in [18]) of the 2D matrix operators of Sato and Nakayashiki [19] with algebraic 2D Bloch-Floquet manifold W . They are non-selfadjoint, but elliptic, and data can be easily chosen to make them real and nonsingular periodic.

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